CRITICAL LATTICES, ELLIPTIC CURVES AND THEIR POSSIBLE DYNAMICS

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We present a combinatorial geometry and dynamical systems framework for the investigation and proof of the Minkowski conjecture about critical determinant of the region $|x|^p + |y|^p < 1$, $p > 1$. The application of the framework may drastically reduce the investigation of sufficiently smooth real functions of many variables. Incidentally, we establish connections between critical lattices, dynamical systems and elliptic curves.

1. Introduction

Voronoï [1] have showed that a lattice is extreme if and only if it is both perfect and eutactic. The notions of the critical lattice are partial case of extreme lattice. Let $\mathcal{D}$ be a set in $n-$dimensional real space $\mathbf{R}^n$. A lattice Λ is the *admissible* for the set $\mathcal{D}$ ($\mathcal{D} - admissible$) if $\mathcal{D} \bigcap \Lambda = \emptyset$ or 0. The infimum $\Delta(\mathcal{D})$ of determinants of all lattices admissible for $\mathcal{D}$ is called *the critical determinant* of $\mathcal{D}$. A lattice Λ is the *critical* if $d(\Lambda) = \Delta(\mathcal{D})$ [2]. Let now $D_p \subset \mathbf{R}^2 = (x, y)$, $p > 1$ be the 2-dimensional region: $|x|^p + |y|^p < 1$. Let $\Delta(D_p)$ be the critical determinant of the region. H. Minkowski [3] have raised a question about critical determinants and critical lattices of regions D_p under varying $p > 1$. Let $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ be two $\mathcal{D}-$admissible lattices each of which contains three pairs of points on the boundary of D_p and such that $(0, 1) \in \Lambda_p^{(0)}$, $(-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}$, (under these conditions the lattices are unique defined). Investigations by H. Minkowski, L. Mordell, C. Davis, H. Cohn, A. Malishev and another researches (please, see the bibliography at [4]) in terms of lattices gave [4]:

Theorem 1.
$$\Delta(D_p) = \begin{cases} d(\Lambda_p^{(1)}), \ 1 < p \leq 2, \ p \geq p_0, \\ d(\Lambda_p^{(0)}), \ 2 \leq p \leq p_0; \end{cases}$$

here p_0 is a real number that is defined unique by conditions $d(\Lambda_{p_0}^{(0)}) = d(\Lambda_{p_0}^{(1)})$, $2,57 \leq p_0 \leq 2,58$.

The Minkowski conjecture about critical determinant of the region D_p can be formulated as the problem of minimization on moduli space $\mathcal{M}$ of admissible lattices of the region D_p (see below). This moduli space is a differentiable manifold. We will call it the *Minkowski's moduli space*. The *tangent bundle* of a differentiable manifold M, denoted by TM, is the union of the tangent spaces at all the points of M. Recall that a *vector field* on a smooth manifold M is a map $F : M \rightarrow TM$ which satisfies $p \circ F = id_M$, where p is the

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natural projection $TM \to M$. By its definition a vector field is a *section* of the bundle TM. There are many classes of dynamical systems on the Minkowski's moduli space. Each vector field defines a dynamical system on M. But we do not consider these dynamical systems in the paper. There are bundles of sufficiently smooth real functions on $\mathcal{M}$ and sections of the bundles define discrete dynamical systems on $\mathcal{M}$. Below we will define interval extension of $\mathcal{M}$ and interval sheaves on the interval extension $\mathcal{I}M$ of $\mathcal{M}$ and interval dynamical systems on $\mathcal{I}M$. Still one class of dynamical systems appear from lattices. After complexification of $\mathbf{R}^2$ and lattices $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ these lattices define two classes of elliptic curves. C. Deninger [5] and authors of the paper [6] have discussed possible dynamics of elliptic curves. At the last section of the paper we will investigate elliptic curves of above-mentioned classes and their possible dynamics. In order to make the paper easy to read we will illustrate definitions of the basic notions by examples.

2. ANALYTICAL FORMULATION OF THE MINKOWSKI'S CONJECTURE

Recall the analytic formulation of Minkowski's conjecture [3], [8], [9]. Below we use notations from [4], [15]. Let

$$\Delta(p,\sigma) = (\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}},$$

be the function defined in the domain

$$D_p : \ \infty > p > 1, \ 1 \leq \sigma \leq \sigma_p = (2^p - 1)^{\frac{1}{p}},$$

of the $\{p, \sigma\}$ plane, where σ is some real parameter; here $\tau = \tau(p, \sigma)$ is the function uniquely determined by the conditions

$$A^p + B^p = 1, \ 0 \leq \tau \leq \tau_p,$$

where

$$A = A(p, \sigma) = (1 + \tau^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}},$$

$$B = B(p, \sigma) = \tau(1 + \sigma^p)^{-\frac{1}{p}} + \sigma(1 + \tau^p)^{-\frac{1}{p}},$$

τ_p is defined by the equation $2(1 - \tau_p)^p = 1 + \tau_p^p, \ 0 \leq \tau_p \leq 1$.

Example 1. Critical lattices for $p = 2$. In the case there are two critical lattices: $\Lambda_2^{(0)}$ and $\Lambda_2^{(1)}$. The lattice $\Lambda_2^{(0)}$ has basis $\omega_1 = (1, 0)$, $\omega_2 = (1/2, \sqrt{3}/2)$. The lattice $\Lambda_2^{(1)}$ has basis $\omega_1 = (-2^{-1/2}, 2^{-1/2})$, $\omega_2 = (\frac{\sqrt{2 - \sqrt{3}}}{2}, \frac{\sqrt{2 + \sqrt{3}}}{2})$.

Example 2. More generally, for $2 \leq p \leq p_0$, the critical lattice $\Lambda_p^{(0)}$ has the basis $\omega_1 = (1, 0)$, $\omega_2 = (1/2, \sigma_p/2)$.

Definition 1. In the notations the surface

$$\Delta - (\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}} = 0,$$

in $3-$dimensional real space with coordinates (σ, p, Δ) is called the *Minkowski's moduli space*.

Minkowski's analytic conjecture (MA): For any real p and σ with conditions $p > 1$, $p \neq 2$, $1 < \sigma < \sigma_p$

$$\Delta(p, \sigma) > \Delta(D_p) = \min(\Delta(p, 1), \Delta(p, \sigma_p)).$$

3. Interval Cellular Covering

Definition 2. For any n and any j, $0 \leq j \leq n$, an $j-dimensional\ interval$ *cell*, or $j-I\text{-}cell$, in $\mathbf{R}^n$ is a subset Ic of $\mathbf{R}^n$ such that (possibly, after permutation of variables) it has the form

$Ic = \{x \in \mathbf{R}^n : \underline{a}_i, \overline{a}_i, r_k \in \mathbf{R} :$
$\underline{a}_i \leq x_i \leq \overline{a}_i, 1 \leq i \leq j,$
$x_{j+1} = r_1, \cdots, x_n = r_{n-j}\}$. Here $\underline{a}_i \leq \overline{a}_i$.

If $j = n$ then we have an $n-$dimensional interval vector. Let $\mathcal{P}$ be the hyperplane that contains Ic. These is the well known fact:

Lemma 1. *The dimension of Ic is equal to the minimal dimension of hyperplanes that contain Ic.*

Let $\mathcal{P}$ be the such hyperplane, $Int\ Ic$ the set of interior points of Ic in $\mathcal{P}$, $Bd\ Ic = Ic \setminus Int\ Ic$. For $m-$dimensional I-cell Ic let d_i be an $(m-1)-$dimensional I-cell from $Bd\ Ic$. Then d_i is called an $(m-1)-$dimensional *face* of the I-cell Ic.

Definition 3. Let D be a bounded set in $\mathbf{R}^n$. By interval cellular covering Cov we will understand any finite set of $n-$dimensional I-cells such that their union contains D and adjacent I-cells are intersected by their faces only. By $|\,Cov\,|$ we will denote the union of all I-cells from Cov.

Let Cov be the interval covering. By its *subdivision* we will understand an interval covering $Cov^{`}$ such that $|\,Cov\,| = |\,Cov^{`}\,|$ and each I-cell from $Cov^{`}$ is contained in an I-cell from Cov. In the paper we will consider mainly bounded horizontal and vertical strips in $\mathbf{R}^2$, their interval coverings and subdivisions.

4. Some Categories and Functors of Interval Mathematics

Let $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) = ([\underline{x}_1, \overline{x}_1], \cdots, [\underline{x}_n, \overline{x}_n]$ be the $n-$dimensional real interval vector with $\underline{x}_i \leq x_i \leq \overline{x}_i$ ("rectangle" or "box"). Let f be a real continuous function of n variables that is defined on $\mathbf{X}$. The *interval evaluation* of f on the interval $\mathbf{X}$ is the interval $[\underline{f}, \overline{f}]$ such that for any $x \in \mathbf{X}$, $f(x) \in [\underline{f}, \overline{f}]$. The interval evaluation is called *optimal* [12] if $\underline{f} = \min f$, and $\overline{f} = \max f$ on the interval $\mathbf{X}$. Let $Of-$ be the optimal interval evaluation of f on $\mathbf{X}$.

Definition 4. The pair $(\mathbf{X}, Of)$ is called the *interval functional element.* If Ef is an interval that contains Of then we will call the pair $(\mathbf{X}, Ef)$ *the extension* of $(\mathbf{X}, Of)$ or $eif-element.$
Let f be the constant signs function on $\mathbf{X}$. If $f > 0$ (respectively $f < 0$) on $\mathbf{X}$ and $Of > 0$ (respectively $Of < 0$) then we will call $(\mathbf{X}, Of)$ *the correct interval functional element* (shortly $c-element$).

More generally we will call *the correct interval functional element* an extension $(\mathbf{X}, Ef)$ of $(\mathbf{X}, Of)$ that has the same sign as Of.

A set of intervals with inclusion relation forms a category $\mathcal{CIP}$ of preorder [13].

Definition 5. A contravariant functor from $\mathcal{CIP}$ to the category of sets is called the interval presheaf.

For a finite set $FS = \{\mathbf{X}_i\}$ of $m-$dimensional intervals in $\mathbf{R}^n$, $m \le n$, the union V of the intervals forms a piecewise-linear manifold in $\mathbf{R}^n$. Let G be the graph of the adjacency relation of intervals from FS. The manifold V is connected if G is a connected graph. In the paper we are considering connected manifolds. Let f be a constant signs function on $\mathbf{X} \in FS$. The set $\{(\mathbf{X}_j, Of)\}$ of $c-$elements (if exists) is called *a constant signs continuation of f on $\{\mathbf{X}_j\}$.* If $\{\mathbf{X}_j\}$ is the maximal subset of FS relatively a constant signs function f then $\{(\mathbf{X}_j, Of)\}$ is called the *constant signs continuation of f on FS.*

5. INTERVAL ITERATIVE PROCESSES

In the section a "dynamical system" is a continuous map $T : X \to X$ (or continuous flow or semiflow $\phi_t : X \to X$) on a compact metric space. The recent investigations of iterated polynomial maps which can be considered as dynamical systems (see, for instance [14] and references in the paper) has provoked an interest in the range of applicability of these methods for transcendental iterated maps and for their interval extensions. Interval mathematics offers a rigorous approach to computer investigation of mathematical models. Interval methods is a kind of numerical methods with automatic result verification. Under the investigation and proof of the Minkowski conjecture we have to compute expressions Δ'_σ, Δ''_{σ^2}, Δ'_p, $\Delta''_{\sigma p}$, $\Delta'''_{\sigma^2 p}$ and their interval extensions. These expressions are represented in terms of a sum of derivatives of "atoms" $s_i = \sigma^{p-i}$, $t_i = \tau^{p-i}$, $a_i = (1 + \sigma^p)^{-i-\frac{1}{p}}$, $b_i = (1 + \tau^p)^{-i-\frac{1}{p}}$, $A = b_0 - a_0$, $B = \tau b_0 + \sigma a_0$, $\alpha_i = A^{p-i}$, $\beta_i = B^{p-i}$ $(i = 0, 1, 2, \ldots)$. Let D be a subdomain of D_p. The domain is covered by rectangles of the form

$$\mathbf{X} = [\underline{p}, \overline{p}; \underline{\sigma}, \overline{\sigma}].$$

Let f be one of mentioned functions. The $eif-$element $(\mathbf{X}, Ef)$ is represented in terms of $\underline{p}$, $\overline{p}$, $\underline{\sigma}$, $\overline{\sigma}$, $\underline{\tau}$, $\overline{\tau}$, ; here the bounds $\underline{\tau}$, $\overline{\tau}$, are obtained with the help of some interval iteration processes. In this section we give formulas

for one of the interval iterative processe on Minkowski's moduli space. Computations of these iterative processes and their interval extensions in various floating points and intervals were produced Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a transcendental map which is a superposition of rational functions, exponential functions and logarithms. We will use the notation of [14] and denote the $n-th$ iterate of a map f by $f^{\circ n}$. Let $\mathbf{IR}^2$ be the set of all intervals in $\mathbf{R}^2$. Let $If : \mathbf{IR}^2 \to \mathbf{IR}$ be an interval extension of f. Let $If^{\circ n}$ be the $n - th$ iterate of the map If. We can consider the evolution of If as (i) evolution of two correlated real dynamical systems $[\underline{I}f^{\circ n}, \overline{I}f^{\circ n}]$, or as (ii) evolution of an interval dynamical system on interval data. The principal considerations concerns case (i). In the case for iterated map

$$[\underline{x}_{k+1},\ \overline{x}_{k+1}] = [\underline{f}(\underline{x}_k, \overline{x}_k, \underline{u}, \overline{u}, \underline{v}, \overline{v}),\ \overline{f}(\underline{x}_k, \overline{x}_k, \underline{u}, \overline{u}, \underline{v}, \overline{v}))]$$

in u, v plane we will describe the evolution of If for various subregions and points of $2-$dimensional region

$$D_p:\ \infty > u > 1,\ 1 \le v \le (2^u - 1)^{\frac{1}{u}},$$

where v is some real parameter. Let us give formulas for interval extension of Minkowski's moduli space: here the bounds $\underline{\tau}$, $\overline{\tau}$, are obtained with the help of the iteration process:

$$\underline{t}_{i+1} = (1 + \underline{t}_i^{\overline{p}})^{\frac{1}{\overline{p}}}(((1 - (1 + \underline{t}_i^{\overline{p}})^{-\frac{1}{\overline{p}}} - (1 + \overline{\sigma}^{\underline{p}})^{-\frac{1}{\underline{p}}})^{\underline{p}})^{\frac{1}{\underline{p}}} - \overline{\sigma}(1 + \overline{\sigma}^{\underline{p}})^{-\frac{1}{\underline{p}}}) \qquad (1)$$

$$\overline{t}_{i+1} = (1 + \overline{t}_i^{\underline{p}})^{\frac{1}{\underline{p}}}(((1 - (1 + \overline{t}_i^{\underline{p}})^{-\frac{1}{\underline{p}}} - (1 + \underline{\sigma}^{\overline{p}})^{-\frac{1}{\overline{p}}})^{\overline{p}})^{\frac{1}{\overline{p}}} - \underline{\sigma}(1 + \underline{\sigma}^{\overline{p}})^{-\frac{1}{\overline{p}}}) \qquad (2)$$

$$i = 0, 1, \cdots$$

As interval computation is the enclosure method, we have to put:

$$[\underline{\tau},\ \overline{\tau}] = [\underline{t}_N,\ \overline{t}_N] \bigcap [\underline{\tau}_0,\ \overline{\tau}_0]\ .$$

N is computed on the last step of the iteration.

For initial values we may take : $[\underline{t}_0,\ \overline{t}_0] = [\underline{\tau}_0,\ \overline{\tau}_0] = [0,\ 0.36]$.

Remark 1. Let $\mathbf{X} = [\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$ be the interval, where the interval iteration process $(1) - (2)$ is computed. Let $p = \frac{\underline{p} + \overline{p}}{2}, \sigma = \frac{\underline{\sigma} + \overline{\sigma}}{2}$. Computations show that for the convergence of the interval iterative processes $(1) - (2)$ it is sufficient that the noninterval inequality $f'_\tau < 1$ is satisfied in the point $[p, \sigma]$.

6. Dynamical systems from critical lattices

6.1. **Algebraic dynamical systems.** Let Λ be a lattice and R its ring of multipliers. So for $\lambda \in R$, $\lambda\Lambda \subseteq \Lambda$ and $\omega \in \Lambda$ an algebraic $\mathbf{Z}-$action $\alpha : n \mapsto \alpha_n$ on Λ is defined by

$$\alpha_n(\omega) = \lambda^n \omega.$$

6.2. **Elliptic curves from critical lattices.** Let Λ be a critical lattice of the domain D_p. After complexification of $\mathbf{R}^2$ the lattice Λ takes form $\Lambda = n\omega_1 + m\omega_2$, $\omega_1, \omega_2 \in \mathbf{C}$, $n, m \in \mathbf{Z}$, $\frac{\omega_1}{\omega_2}$ is not a real number.

Example 3. After complexification the basis of the critical lattice $\Lambda_2^{(0)}$ has the form $\omega_1 = 1$, $\omega_2 = 1/2 + \frac{\sqrt{3}}{2}i$.

Respectively for $2 \le p \le p_0$, the lattice $\Lambda_p^{(0)}$ has the basis $\omega_1 = 1$, $\omega_2 = 1/2 + \frac{\sigma_p}{2}i$.

Let Λ be as above. For $\alpha \in \Lambda$ construct invariants of Λ :

$$\sideset{}{'}\sum_{\alpha \in \Lambda} \frac{1}{\alpha^n} = \sum_{\alpha \ne 0,\, \alpha \in \Lambda} \frac{1}{\alpha^n}.$$

Let

$$c_n = \sideset{}{'}\sum_{\alpha \in \Lambda} \frac{1}{\alpha^{2n}}.$$

There is well known

Lemma 2. *If $t \in \mathbf{R}, t > 2$, then the series $\sum_{\alpha \in \Lambda}' \frac{1}{\alpha^t}$, converges absolutely.*

The Weierstrass elliptic function is defined as expression

$$\frac{1}{z^2} + \sideset{}{'}\sum \left(\frac{1}{(z+\alpha)^2} - \frac{1}{\alpha^2} \right).$$

Each Weierstrass elliptic function defines the field $\mathcal{K}_\Lambda$ of elliptic functions. Its model is the elliptic curve in the Weierstrass normal form:

$$y^2 = 4x^3 - 60c_2 x - 140c_3.$$

6.3. **Dynamical systems from the elliptic curves.** The *Julia set* of a rational function $f \in \mathbf{C}(z)$ is the closure of the union of its repelling cycles [14]. Let $f(z)$ be the rational function that is constructed under division of points of an elliptic curve over $\mathbf{C}$[16]. The Julia set of the rational functions is by the result of S. Lattés [17] the whole sphere $\overline{\mathbf{C}}$.

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